MTH 301: Group Theory

Homework I

(Due 19/08)

1. In each of the following, prove that $(G, \star)$ is a group.
   (a) $(\mathbb{C} - \{0\}, \cdot)$, where $\cdot$ denotes the complex product.
   (b) $(\mathrm{GL}(n, \mathbb{R}), \cdot)$, where $\cdot$ denotes the matrix product.
   (c) $(\mathbb{Z}_n, +)$
   (d) $(S_n, \circ)$
   (e) $(G, +)$, where $G = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.
   (f) $(G, \cdot)$, where $G = \{z \in \mathbb{C} \mid z^n = 1, \text{ for some fixed } n \in \mathbb{Z}^+\}$.
2. In each of the following, determine why $(G, \star)$ is not a group
   (a) $(\mathbb{Z}, -)$
   (b) $(\mathbb{Q} - \{0\}, \div)$
   (c) $(\mathbb{R}, \star)$, where $a \star b = a + b + ab$.
3. Let G be a group and $g \in G$. Then:
   (a) Show that $g^2 = 1$ if and only if $o(g) = 1$ or 2.
   (b) $o(g) = o(g^{-1})$.
   (c) If $o(g) = n$, then for some $n \in \mathbb{Z}^+$, $g^{-1} = g^{n-1}$.
   (d) $o(g) = o(hgh^{-1})$ for any $h \in G$.
4. Let G be a group. Prove that if $g^2 = 1$ for every $g \in G$, then G is abelian.